Exercise 6

Verify the Cauchy–Schwarz inequality and the triangle inequality for the vectors in Exercises 3 to 6.

$$\mathbf{x} = (1, 0, 0, 1),\ \mathbf{y} = (-1, 0, 0, 1)$$

Solution

Cauchy–Schwarz Inequality

Check the Cauchy–Schwarz inequality $|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\|\|\mathbf{y}\|$ for the given vectors.

$$|\mathbf{x} \cdot \mathbf{y}| = |(1)(-1) + (0)(0) + (0)(0) + (1)(1)| = |0| = 0$$

$$\|\mathbf{x}\| = \sqrt{1^2 + 0^2 + 0^2 + 1^2} = \sqrt{2}$$

$$\|\mathbf{y}\| = \sqrt{(-1)^2 + 0^2 + 0^2 + 1^2} = \sqrt{2}$$

As a result,

$$|\mathbf{x} \cdot \mathbf{y}| = 0 \leq \sqrt{2}\sqrt{2} = \|\mathbf{x}\|\|\mathbf{y}\|,$$

which means the Cauchy–Schwarz inequality is satisfied.

Triangle Inequality

Now check the triangle inequality $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ for the given vectors.

$$\mathbf{x} + \mathbf{y} = (1, 0, 0, 1) + (-1, 0, 0, 1) = (0, 0, 0, 2)$$

$$\|\mathbf{x} + \mathbf{y}\| = \sqrt{0^2 + 0^2 + 0^2 + 2^2} = 2$$

$$\|\mathbf{x}\| = \sqrt{1^2 + 0^2 + 0^2 + 1^2} = \sqrt{2}$$

$$\|\mathbf{y}\| = \sqrt{(-1)^2 + 0^2 + 0^2 + 1^2} = \sqrt{2}$$

As a result,

$$\|\mathbf{x} + \mathbf{y}\| = 2 \leq \sqrt{2} + \sqrt{2} = \|\mathbf{x}\| + \|\mathbf{y}\|,$$

which means the triangle inequality is satisfied.